Hicksian Demands and Expenditure Function for Quasilinear Preferences

Given the following utility function $u(x_1, x_2) = x_1 + ln(x_2)$:

1. Obtain the Hicksian demands.
2. Obtain the expenditure function.
3. Obtain the Hicksian demands without solving the expenditure minimization problem.

Solutions

1. Construct the Lagrangian for minimization:

$$L = p_1x_1 + p_2x_2 + \lambda(\bar{U} - x_1 - ln(x_2))$$

The first-order conditions are:

$$L'_{x_1} = p_1 - \lambda = 0$$

$$L'_{x_2} = p_2 - \frac{\lambda}{x_2} = 0$$

$$L'_{\lambda} = \bar{U} - x_1 - ln(x_2) = 0$$

From the first two equations, solve for λ and equate:

$$p_1 = p_2x_2$$

Therefore:

$$x_2^h = \frac{p_1}{p_2}$$

Now solve for x_1:

$$\bar{U} - x_1 - ln(\frac{p_1}{p_2}) = 0$$

$$x_1^h = \bar{U} - ln(\frac{p_1}{p_2})$$

If p_1 is very high or $\bar{U}$ is very low, then the optimal quantities of x_1 should be zero. This leads us to piecewise Hicksian demands:

$$x_1^h = \begin{cases} \bar{U} - ln(\frac{p_1}{p_2}) & \text{if } \bar{U} \geq ln(\frac{p_1}{p_2}) \\ 0 & \text{if } \bar{U} < ln(\frac{p_1}{p_2}) \end{cases}$$

And therefore, this leads us to the following piecewise function for x_2:

$$x_2^h = \begin{cases} \frac{p_1}{p_2} & \text{if } \bar{U} \geq ln(\frac{p_1}{p_2}) \\ e^{\bar{U}} & \text{if } \bar{U} < ln(\frac{p_1}{p_2}) \end{cases}$$

2. The expenditure function is also piecewise:

$$E(p_1, p_2, \bar{U}) = \begin{cases} p_1[\bar{U} - ln(\frac{p_1}{p_2})] + p_1 & \text{if } \bar{U} \geq ln(\frac{p_1}{p_2}) \\ p_2e^{\bar{U}} & \text{if } \bar{U} < ln(\frac{p_1}{p_2}) \end{cases}$$

3. To do this, we use Shepard's Lemma (assuming an interior solution). First, we re-express the expenditure function:

$$E = p_1\bar{U} - p_1ln(p_1) + p_1ln(p_2) + p_1$$

$$\frac{\partial E}{\partial p_1} = x_1^h = \bar{U} - [ln(p_1) + \frac{p_1}{p_1}] + ln(p_2) + 1 = \bar{U} - (ln(p_1) - ln(p_2)) = \bar{U} - ln(\frac{p_1}{p_2})$$

$$\frac{\partial E}{\partial p_2} = x_2^h = \frac{p_1}{p_2}$$

Now assume a corner solution:

$$\frac{\partial E}{\partial p_1} = x_1^h = 0$$

$$\frac{\partial E}{\partial p_2} = x_2^h = e^{\bar{U}}$$